

1. Find the Eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

[A.U M/J 2011]

Soln:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

The characteristic eqn of A is $|A - \lambda I| = 0$

i.e., $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = \text{sum of the main diagonal elements} = (1) + (-1) = 0.$

$S_2 = |A| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

 $\therefore$ the characteristic eqn is

$\lambda^2 - 0\lambda + (-4) = 0$

$\lambda^2 - 4 = 0$

$\lambda^2 = 4$

$\lambda = \pm 2$

 $\therefore$ The Eigen values are $-2, 2$.

2. Find the Eigenvector corresponding to the Eigenvalue 1 of the matrix

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

[A.U CBT J/F 2009]

Soln:

Eigen value = 1.

To find Eigen vector,

$(A - \lambda I) X = 0.$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 1$ we get.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (1)}$$

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (2)}$$

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (3)}$$

$$(1) = (2) = (3)$$

$$x_1 + 2x_2 + x_3 = 0$$

Put $x_1 = 0$ we get

$$2x_2 + x_3 = 0$$

$$2x_2 = -x_3$$

$$\frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore \text{Eigenvector} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

3. The product of two Eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16.
Find the third Eigenvalue. [A.U Jan 2012, 2014] [D/T 2016 R-08].

Soln:

$$\lambda_1 \lambda_2 = 16$$

Wkt, $\lambda_1 \lambda_2 \lambda_3 = |A|$ [$\because$ Product of Eigen values = determinant of A]

$$\therefore \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+3) + 2(2-6) \Rightarrow 6(8) + 2(-3) + 2(-4)$$

$$\Rightarrow 48 - 6 - 8$$

$$\Rightarrow 32$$

$$\lambda_1 \lambda_2 \lambda_3 = 32$$

$$16 \cdot \lambda_3 = 32$$

$$\lambda_3 = \frac{32}{16} = 2$$

$$\boxed{\lambda_3 = 2}$$

4. Two Eigenvalues of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3 and 0. What is the third Eigenvalue? What is the product of the Eigenvalues of A? [A.U P.T O/N 2002] [A.U A/M 2017 R-13]

Soln:

$$\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = ?$$

Wkt, sum of the Eigenvalues = sum of the Main diagonal elements.

$$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 0 + \lambda_3 = 18$$

$$\boxed{\lambda_3 = 18 - 3 = 15}$$

$$\text{Product of the Eigen values} = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 3 \cdot 0 \cdot 15 = 0.$$

5. Find the sum and product of all the Eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$$

[A.U CBT J/J 2010, A/M 2011, M/J 2010]

Soln:

$$\text{Sum of the eigen values} = 1 + 0 + (-3) = -2.$$

$$\text{Product of the eigen values} = |A|.$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{vmatrix}$$

$$= 1(0 + 3) - 2(-3 + 6) - 2(-1 + 0)$$

$$= 3 - 6 + 2 = -1.$$

$$\boxed{|A| = -1}$$

6. Prove that if λ is an Eigenvalue of a matrix A , then $\frac{1}{\lambda}$ is the Eigenvalue of A^{-1} , ($\lambda \neq 0$) [A.U. Mar 1996, May 1997, M/J 2006, N/D 2014 R-13]

[OR]

If λ is an Eigen value of a matrix A , what can you say about the Eigenvalue of matrix A^{-1} . Prove your statement, ($\lambda \neq 0$).

Proof:

If X be the Eigenvector corresponding to λ ,
then $AX = \lambda X \rightarrow \text{①}$.

Premultiplying both sides by A^{-1} , we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$$\text{i.e., } A^{-1}X = \frac{1}{\lambda}X \rightarrow \text{This is same form of ①}$$

$\therefore \frac{1}{\lambda}$ is the Eigenvalue of A^{-1} .

7. The Eigenvalues of a null matrix are zero. [A.U. A/M 2018 R-17]

Proof:

If A is a null square matrix of order 3 then its characteristic equation is

$$(-1)^3 \lambda^3 = 0 \Rightarrow \lambda^3 = 0$$

$\therefore$ the Eigenvalues of A are 0, 0, 0.

In general the characteristic equation of a null square matrix of order n is $(-1)^n \lambda^n = 0$.

$$\text{i.e., } \lambda^n = 0$$

Therefore the Eigenvalues of A are zero.

8. Find the Eigenvalues of A^{-1} if the matrix A is $\begin{bmatrix} 2 & 5 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ ③

[A U J/J 2010, A/M 2011, M/J 2010]

Soln,

Qn. A is a triangular matrix.

Hence the Eigenvalues are 2, 3, 4.

By the property the Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

9. The Eigenvalues of a real symmetric matrix are real numbers.
[Nov 2001] [M/J 2014 R-13]

Proof:

Let λ be an Eigenvalue of the real symmetric matrix A and x be the corresponding Eigenvector.

$$\text{Thus, } Ax = \lambda x \quad \text{--- (1)}$$

$$\overline{A} \overline{x} = \overline{\lambda} \overline{x}$$

$$A \overline{x} = \overline{\lambda} \overline{x} \quad [\because \overline{A} = A, \text{ as } A \text{ is real}]$$

$$(A \overline{x})^T = (\overline{\lambda} \overline{x})^T$$

$$(\overline{x})^T A^T = \overline{\lambda} (\overline{x})^T$$

$$(\overline{x})^T A = \overline{\lambda} (\overline{x})^T \quad [\because A^T = A \text{ as } A \text{ is symmetric}]$$

$$(\overline{x})^T Ax = \overline{\lambda} (\overline{x})^T x$$

$$(\overline{x})^T \lambda x = \overline{\lambda} (\overline{x})^T x \quad [\text{by (1) } Ax = \lambda x]$$

$$\lambda (\overline{x})^T x = \overline{\lambda} (\overline{x})^T x$$

$$\Rightarrow (\lambda - \overline{\lambda}) (\overline{x})^T x = 0$$

$$\Rightarrow (\lambda - \overline{\lambda}) = 0 \quad (\overline{x})^T x \neq 0$$

$$\boxed{\lambda = \overline{\lambda}}$$

$\therefore \lambda$ is real.

10. If 2, -1, -3 are the Eigenvalues of the matrix A, then find the Eigenvalues of the matrix $A^2 - 2I$. [AU M/J 2014 R-13]

Soln:

Given: The Eigenvalues of A are 2, -1, -3

The Eigenvalues of $A^2 - 2I$ are

$$2^2 - 2(1) = 2$$

$$(-1)^2 - 2(1) = -1$$

$$(-3)^2 - 2(1) = 9 - 2 = 7.$$

11. Find the constants a & b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 & -2 as its Eigenvalues. [A.U. A/M 2011] [A.U. A/M 2017 R-13].

Soln:

Sum of the Eigenvalues of a matrix = sum of the elements of the main diagonal.

$$(3) + (-2) = a + b$$

$$3 - 2 = a + b$$

$$S_1 = \boxed{a + b = 1} \quad \text{--- (1)}$$

Product of the Eigenvalues = determinant of the matrix

$$(3)(-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix}$$

$$-6 = ab - 4$$

$$-6 + 4 = ab$$

$$S_2 = \boxed{ab = -2}$$

$$x^2 - S_1 x + S_2 = 0$$

$$x^2 - (a+b)x + ab = 0$$

$$x^2 - 1 \cdot x - 2 = 0$$

$$x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$$

$$x = -1, x = 2$$

$$\boxed{a = 2, b = -1 \text{ (or) } a = -1, b = 2}$$

$$\lambda = \begin{matrix} -2 \\ -1 \\ 1 \end{matrix}$$

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12. Find the Eigenvalue of a matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ corresponding to the Eigenvector $\begin{bmatrix} -4 & -2 & 4 \end{bmatrix}^T$. [A.U. A/M 2017 R-08]

Solution:

$$\text{W.K.T } (A - \lambda I)X = 0.$$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$14 - 2\lambda - 2 = 0$$

$$12 - 2\lambda = 0$$

$$+2\lambda = +12$$

$$\boxed{\lambda = 6}$$

Hence the corresponding Eigenvalue $= 6$.

13. State Cayley-Hamilton theorem.

Every square matrix A satisfies its own characteristic equation.

14. If $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$ express A^3 in terms of A & I using Cayley Hamilton theorem [NOV 2001] [A.U. CBT D/S 2011, A/M 2011].

Soln:

The characteristic equation of the given matrix is $|A - \lambda I| = 0$.

$$\text{i.e., } \left| \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 4 & 5-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(5-\lambda) - 0 = 0$$

$$(1-\lambda)(5-\lambda) = 0.$$

$$5-\lambda - 5\lambda + \lambda^2 = 0.$$

$$\lambda^2 - 6\lambda + 5 = 0$$

By Cayley-Hamilton theorem,

[Every square matrix A satisfies its own characteristic equation]

$$\text{i.e., } A^2 - 6A + 5I = 0$$

$$A^2 = 6A - 5I$$

Multiply A on b.s.

$$A^3 = 6A^2 - 5A$$

$$= 6[6A - 5I] - 5A$$

$$= 36A - 30I - 5A$$

$$\boxed{A^3 = 31A - 30I}$$

15. Use Cayley-Hamilton theorem to find $A^4 - 8A^3 + 12A^2$ when $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

[A.U N/D 2010]

Soln:

The characteristic eqn of the given matrix is

$$|A - \lambda I| = 0.$$

$$\text{i.e., } \left| \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0.$$

$$\left| \begin{matrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{matrix} \right| = 0$$

$$(5-\lambda)(3-\lambda) - 3 = 0$$

$$15 - 5\lambda - 3\lambda + \lambda^2 - 3 = 0.$$

$$\lambda^2 - 8\lambda + 12 = 0$$

By Cayley-Hamilton theorem,

$$(1) \Rightarrow A^2 - 8A + 12I = 0.$$

$$\times A^2 \Rightarrow A^4 - 8A^3 + 12A^2 = 0$$

16. S.T the matrix $P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal. [A.U. 1/M 2009, J/J 2010, D/J 2011] (5)

Soln:

Condition for orthogonal matrix is $|P| = \pm 1$

$$|P| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$|P| = \cos^2\theta + \sin^2\theta = 1$$

$\therefore$ Given matrix is orthogonal.

17. Can $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be diagonalized? why? [A.U. Jan 2010, Jan 2012]

Soln: The given matrix 'A' is a real symmetric and non-singular matrix. Hence A can be diagonalised.

18. What is diagonalisation of a symmetric matrix? [A.U. M/J 2010]

Soln: If a square matrix of order n has n linearly independent Eigenvalues, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

19. Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$ [A.U. Nov. 2011, M/J 2013].

Soln:

$$A = \begin{bmatrix} \text{coeff } x^2 & \frac{1}{2} \text{coeff } xy & \frac{1}{2} \text{coeff } xz \\ \frac{1}{2} \text{coeff } xy & \text{coeff } y^2 & \frac{1}{2} \text{coeff } yz \\ \frac{1}{2} \text{coeff } xz & \frac{1}{2} \text{coeff } yz & \text{coeff } z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

20. Give the nature of a quadratic form whose matrix is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

[A.U. A/M 2015 Q-13]

Soln:

The Eigenvalues of the given matrix are $-1, -1, -2$.

All the Eigenvalues are negative numbers.

$\Rightarrow$ The nature of the quadratic form is negative definite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(6)

UNIT-II

VECTOR CALCULUS

1. What is the greatest rate of increase of $\phi = xyz^2$ at $(1, 0, 3)$?

[A.U N/D 2005] [A.U A/M 2017].

Soln:

Given $\phi = xyz^2$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$\nabla \phi_{(1,0,3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$$\therefore \text{Greatest rate of increase} = |\nabla \phi| = \sqrt{9^2} = 9.$$

2. P.T $\nabla f(r) = \frac{f'(r)}{r} \vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ [A.U April 2017 R-15 U.D]

Soln:

$$\nabla f(r) = \sum \vec{i} \frac{\partial}{\partial x} f(r) = \sum \vec{i} f'(r) \frac{\partial r}{\partial x}$$

$$= \sum \vec{i} f'(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{f'(r)}{r} \vec{r}$$

3. Find a unit vector normal to the surface $x^2 + y^2 - z = 1$ at $(1, 1, 1)$

[A.U May 1996, A/M 2017]

Soln:

Given $\phi = x^2 + y^2 - z - 1$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla \phi_{(1,1,1)} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla \phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

4. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2 y^2 z^4$ maximum? [A.U. N/D 2008, M/M 2015 R-13, AU APRIL 2016 P-15]

Soln:

Given $\phi = x^2 y^2 z^4$.

$$\begin{aligned}\nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \vec{i} [2xy^2z^4] + \vec{j} [2x^2yz^4] + \vec{k} [4x^2y^2z^3] \\ (\nabla \phi)_{(3,1,-2)} &= \vec{i} [2(3)(1)(16)] + \vec{j} [2(9)(1)(16)] + \vec{k} [4(9)(1)(-8)] \\ &= 96\vec{i} + 288\vec{j} - 288\vec{k}\end{aligned}$$

The directional derivative is maximum in the direction $\nabla \phi$.

5. Find the divergence of the vector field. [A.U. M/J 2016 R-08]
 $\vec{A} = (xyz)\vec{i} + (3x^2y)\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point $(2, -1, 1)$.

Soln:

$$\begin{aligned}\nabla \cdot \vec{A} &= \frac{\partial}{\partial x} (xyz) + \frac{\partial}{\partial y} (3x^2y) + \frac{\partial}{\partial z} (xz^2 - y^2z) \\ &= yz + 3x^2 + 2xz - y^2\end{aligned}$$

$$\begin{aligned}(\nabla \cdot \vec{A})_{(2,-1,1)} &= (-1)(1) + 3(2)^2 + 2(2)(1) - (-1)^2 \\ &= -1 + 12 + 4 - 1 = 14\end{aligned}$$

6. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find $\text{div}(\text{curl } \vec{F})$. [May 1996
 AU A/M 2018 R-17]

Soln:

$$\begin{aligned}\text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix} \\ &= \vec{i} \frac{\partial}{\partial y} (z^3) - \frac{\partial}{\partial z} (y^3) - \vec{j} \frac{\partial}{\partial x} (z^3) - \frac{\partial}{\partial z} (x^3) \\ &\quad + \vec{k} \frac{\partial}{\partial x} (y^3) - \frac{\partial}{\partial y} (x^3)\end{aligned}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}.$$

$$\therefore \text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0.$$

7. Find the constants a, b, c so that

$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Soln.

[A.U M/J 2009, A/M 2017 P-13

A/M 2018 P-17]

$$\text{Given } \nabla \times \vec{F} = \vec{0}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}[c+1] - \vec{j}[4-a] + \vec{k}[b-2] = 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\begin{aligned} \text{i.e. } c+1 &= 0 & 4-a &= 0 & b-2 &= 0 \\ c &= -1 & a &= 4 & b &= 2 \end{aligned}$$

8. Find 'a' such that $(3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal. [A.U M/J 2006, M/J 2012, A/M 2015 P-8, P-13]

Soln.

$$\text{Given } \vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}.$$

$$\nabla \cdot \vec{F} = 0$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k} \\ &= \frac{\partial}{\partial x} (3x-2y+z) + \frac{\partial}{\partial y} (4x+ay-z) + \frac{\partial}{\partial z} (x-y+2z) \end{aligned}$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\Rightarrow 5 + a = 0$$

$$\boxed{a = -5}$$

9. If $\vec{A}$ & $\vec{B}$ are irrotational, then p.t $\vec{A} \times \vec{B}$ is solenoidal. [A.U M/J 2013]

Soln:

Qn, $\vec{A}$ & $\vec{B}$ are irrotational

$$(1) \nabla \times \vec{A} = \vec{0} \text{ \& } \nabla \times \vec{B} = \vec{0}$$

$$\begin{aligned} \text{w.k.t, } \nabla \cdot (\vec{A} \times \vec{B}) &= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} \\ &= \vec{0} \cdot \vec{B} - \vec{0} \cdot \vec{A} \\ &= 0 - 0 = 0 \end{aligned}$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

10. Determine whether $\vec{F} = (2xy + z^2)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. [A.U A/M 2008]

Soln:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \vec{0}$.

$$\therefore \text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (3xz^2) - \frac{\partial}{\partial z} (x^2) \right] - \vec{j} \left[\frac{\partial}{\partial x} (3xz^2) - \frac{\partial}{\partial z} (2xy + z^2) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (2xy + z^2) \right]$$

$$= \vec{i} [0 - 0] + \vec{j} [3z^2 - 2z] + \vec{k} [2x - 2x]$$

$$= \vec{0} - \vec{j} [3z^2 - 2z] + \vec{0}$$

$$\nabla \times \vec{F} \neq \vec{0}$$

$\therefore \vec{F}$ is not a conservative field.

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11. State the physical interpretation of the line integral $\int_A^B \vec{F} \cdot d\vec{r}$

Soln:

[A.U. A/M 2011]

Physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, in displacing a particle from A to B along the curve C.

12. Define Volume Integrals:

The Volume integral of $F(x, y, z)$ over a region enclosing a volume V is given by $\iiint_V F(x, y, z) dv$ or $\iiint_V F(x, y, z) dx dy dz$.

13. Find the area of a circle of radius 'r' using Green's theorem.

Soln:

[A.U. N/D 2004, 2005] [A.U. A/M 2017 R-13]

By Green's theorem, we know that,

$$\text{Area enclosed by } C = \frac{1}{2} \int_C x dy - y dx.$$

The parametric equation of a circle of radius 'r' is

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dx = -r \sin \theta d\theta$$

$$dy = r \cos \theta d\theta$$

Where $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} \text{Area of the circle} &= \frac{1}{2} \int_0^{2\pi} (r \cos \theta) (r \cos \theta d\theta) - (r \sin \theta) (-r \sin \theta d\theta) \\ &= \frac{1}{2} \int_0^{2\pi} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \frac{1}{2} r^2 \int_0^{2\pi} d\theta = \frac{r^2}{2} (\theta)_0^{2\pi} \\ &= \frac{r^2}{2} (2\pi - 0) = \pi r^2 \\ &= \pi r^2 // \end{aligned}$$

14. State Green's theorem in plane.

Statement:

If $u, v, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous and single valued functions in the region R enclosed by the curve C , then

$$\int_C (u dx + v dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy.$$

Corollary (1)

If $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, then the value of the integral $\int_C (u dx + v dy)$ is independent of the path of integration.

Corollary (2)

If R is a region bounded by a simply closed curve C then the area of R is given by $\frac{1}{2} \int_C (x dy - y dx)$.

15. Using Green's theorem, P.T. that the area enclosed by a simple closed curve C is $\frac{1}{2} \int_C x dy - y dx$ [A.U M/J 2007, N/D 2011, M/J 2016, AP 2016 R-15 J].

Soln:

w.k.t, Green's theorem is

$$\int_C (u dx + v dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy.$$

$$\text{Here, } v = \frac{x}{2} \quad u = -\frac{y}{2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \quad \frac{\partial u}{\partial y} = -\frac{1}{2}$$

$$(1) \Rightarrow \int_C \frac{x}{2} dy - \frac{y}{2} dx = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dx dy$$

$$\frac{1}{2} \int_C x dy - y dx = \iint_R dx dy.$$

Q)

16. If S is any closed surface enclosing volume V and

$$\vec{F} = (ax\vec{i} + by\vec{j} + cz\vec{k}) \quad \text{P.T.} \iint_S \vec{F} \cdot \vec{n} \, ds = (a+b+c)V. \quad [\text{A.U. A/M. 2004}]$$

Soln:

Gauss Divergence theorem is

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, ds &= \iiint_V \nabla \cdot \vec{F} \, dv. \\ &= \iiint_V \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (ax\vec{i} + by\vec{j} + cz\vec{k}) \, dv. \\ &= \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) \, dv. \\ &= \iiint_V (a+b+c) \, dv. \\ &= (a+b+c) \iiint_V dv. \\ &= (a+b+c)V. \end{aligned}$$

17. State Gauss divergence theorem. [A.U. M/J 2010, N/D 2012, M/J 2012].

Soln:

The surface integral of the normal component of a vector function taken over a closed surface S , enclosing a volume V is equal to the volume integral of the divergence of $\vec{F}$, taken throughout the volume V .

In other words, If the vector function $\vec{F}$ has continuous first partial derivatives in V , then

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div } \vec{F} \, dv.$$

$$\text{i.e., } \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv.$$

18. If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, a, b, c are constants, show that
$$\iint_S \vec{F} \cdot \hat{n} \, ds = \frac{4\pi}{3} (a+b+c)$$
 where S is the surface of a unit sphere. [A.U A/M 2008]

Proof:

w.k.t, Gauss's divergence theorem,

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} \, ds &= \iiint_V \nabla \cdot \vec{F} \, dV \\&= \iiint_V \left[\frac{\partial}{\partial x} (ax) + \frac{\partial}{\partial y} (by) + \frac{\partial}{\partial z} (cz) \right] dV \\&= \iiint_V (a+b+c) dV \\&= (a+b+c) V = (a+b+c) \frac{4}{3} \pi (1)^3\end{aligned}$$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \frac{4\pi}{3} (a+b+c).$$

19. State Stoke's theorem.

Statement:

The surface integral of the normal component of the curl of a vector function F , over an open surface S , is equal to the line integral of the tangential component of F around the closed curve C bounding S .

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds.$$

(10)

20. Evaluate by Stoke's theorem $\int_C (e^x dx + 2y dy - dz)$, where C is the curve $x^2 + y^2 = 4$, $z = 2$. [A.U M/J 2007]

Solution:

Stoke's theorem is $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds \rightarrow \textcircled{1}$

Qn, $\vec{F} \cdot d\vec{r} = e^x dx + 2y dy - dz$

$$\therefore \vec{F} = e^x \vec{i} + 2y \vec{j} - \vec{k}$$

$$d\vec{r} = \vec{i} dx + \vec{j} dy + \vec{k} dz$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (-1) - \frac{\partial}{\partial z} (2y) \right]$$

$$- \vec{j} \left[\frac{\partial}{\partial x} (-1) - \frac{\partial}{\partial z} (e^x) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (2y) - \frac{\partial}{\partial y} (e^x) \right]$$

$$= \vec{i} (0 - 0) - \vec{j} (0 - 0) + \vec{k} (0 - 0)$$

$$\text{Curl } \vec{F} = \vec{0}$$

$$(1) \Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

$$(2) \int_C (e^x dx + 2y dy - dz) = 0 //$$

① Check whether $w = \bar{z}$ is analytic everywhere. [Anna Nov 2001]

(OR)

Prove that $f(z) = \bar{z}$ is nowhere analytic [AU CBT 3/F 2009]

(OR)

Verify whether $f(z) = \bar{z}$ is analytic function or not.

[A.U N/D 2010, N/D 2012, M/J 2014 R-13].

Soln:

$$\text{Let } w = f(z) = \bar{z}$$

$$u + iv = x - iy$$

$$u = x \quad v = -y$$

$$u_x = 1 \quad v_x = 0$$

$$u_y = 0 \quad v_y = -1$$

$u_x \neq v_y$ at any point $P(x, y)$.

Hence, C-R equations are not satisfied.

$\therefore$ The function $f(z)$ is nowhere analytic.

② Verify $f(z) = z^3$ is analytic or not. [A.U N/D 2014 R-13].

Soln:

$$\text{Given } f(z) = z^3$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$u + iv = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3(x)(iy)^2 + (iy)^3$$

$$= x^3 + 3x^2y i - 3xy^2 - y^3 i$$

$$= (x^3 - 3xy^2) + (3x^2y - y^3) i$$

Equating the real and imaginary parts, we get

$$u = x^3 - 3xy^2$$

$$v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$u_x = v_y$$

$$v_x = u_y$$

$u, f(z)$ satisfies C-R equations.

Moreover u_x, u_y, v_x, v_y exist and are continuous

Therefore, by sufficient conditions $f(z)$ is analytic.

8) P.T $w = \sin 2z$ is an analytic function.

Soln:

$$\text{Let } w = f(z) = \sin 2z.$$

$$u + iv = \sin 2(x + iy)$$

$$= \sin (2x + i2y)$$

$$= \sin 2x \cos(i2y) + \cos 2x \sin i2y$$

$$= \sin 2x \cosh 2y + i \cos 2x \sinh 2y$$

Equating real & imaginary parts, we get,

$$u = \sin 2x \cosh 2y$$

$$v = \cos 2x \sinh 2y$$

$$u_x = 2 \cos 2x \cosh 2y$$

$$v_x = -2 \sin 2x \sinh 2y$$

$$u_y = 2 \sin 2x \sinh 2y$$

$$v_y = 2 \cos 2x \cosh 2y$$

$$u_x = v_y \quad \text{and} \quad -v_x = u_y$$

C-R equations are satisfied.

Also the four partial derivatives are continuous.

Hence the function is analytic.

4. State sufficient conditions for Analytic function. (12) [AUM/J 2008 CBT]

Soln: If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D .

5. State any two properties of analytic function. [AUM/J 2009]

Soln: Prop 1: The real and imaginary parts of an analytic function $w = u + iv$ satisfy the Laplace equation in two dimensions viz.

$$\nabla^2 u = 0 \text{ and } \nabla^2 v = 0.$$

(2) An analytic function with constant modulus is constant.

(3) An analytic function whose real (or) imaginary part is constant, must itself be a constant.

6. Show that an analytic function with constant imaginary part is constant.

Soln: Given $v = a$ (constant)

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

$$f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = 0 + i0 = 0.$$

C.R equations $u_x = v_y, u_y = -v_x$.

$$f'(z) = 0.$$

ie, $f(z) = \text{constant}$.

7. S.T $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic at $z=0$. [May 1996, AP 2016 U.D R-15, D15/J16 R-085].

Soln:

$$\text{Let } z = x + iy$$

$$\bar{z} = x - iy$$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$\text{i.e., } f(z) = |z|^2 = (x^2 + y^2) + i0$$

$$u = x^2 + y^2$$

$$v = 0$$

$$u_x = 2x$$

$$v_x = 0$$

$$u_y = 2y$$

$$v_y = 0.$$

So, the C-R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at $z=0$.

So, $f(z)$ may be differentiable only at $z=0$.

Now, $u_x = 2x$, $u_y = 2y$, $v_x = 0$ and $v_y = 0$ are continuous everywhere and a particular at $(0,0)$.

Hence, the sufficient conditions for differentiability are satisfied by $f(z)$ at $z=0$.

So, $f(z)$ is differentiable at $z=0$ only and is not analytic there.

8. If $f(z) = r^2(\cos 2\theta + i \sin \theta)$ is analytic, then find the value of

Soln:

$$\text{Given } f(z) = r^2 \cos 2\theta + i r^2 \sin \theta$$

$$u + iv = r^2 \cos 2\theta + i r^2 \sin \theta$$

$$\Rightarrow u = r^2 \cos 2\theta$$

$$v = r^2 \sin \theta$$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta$$

$$\frac{\partial v}{\partial r} = 2r \sin \theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$$

$$\frac{\partial v}{\partial \theta} = r^2 \cos \theta.$$

(13)

Given : $f(z)$ is analytic.

$\therefore$ C-R equation

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{1}{r} \frac{\partial u}{\partial \theta} & \frac{\partial v}{\partial r} &= \frac{1}{r} \frac{\partial u}{\partial \theta} \\ 2r \cos 2\theta &= -\frac{1}{r} p r^2 \cos \theta & 2r \sin \theta &= \left(-\frac{1}{r}\right) (-2r^2 \sin 2\theta) \\ 2 \cos 2\theta &= -p \cos \theta & 2 \sin \theta &= 2 \sin 2\theta \\ \Rightarrow p &= -2. & \Rightarrow p &= 2. \end{aligned}$$

Both the cases $p = -2$ $\therefore p = 2$

9. S.T xy^2 can not be the real part of an analytic function.

Examine whether the function $u = xy^2$ can be a real part of an analytic function

[A.U X/M 2018 R-17] [Dec 2009]

Soln:

If a function is analytic its real part must be a harmonic function.

$$\text{Let } u = xy^2$$

$$u_x = y^2$$

$$u_y = 2xy$$

$$u_{xx} = 0$$

$$u_{yy} = 2x$$

$$u_{xx} + u_{yy} = 2x \neq 0 \text{ (except at } x=0)$$

As u is not harmonic, it can not be the real part of an analytic function.

10. Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

Soln:

$$\text{Given : } u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

To prove : u is harmonic.

$$\text{i.e., to prove : } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\begin{array}{l|l} \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x & \frac{\partial u}{\partial y} = -6xy - 6y \\ \frac{\partial^2 u}{\partial x^2} = 6x + 6 & \frac{\partial^2 u}{\partial y^2} = -6x - 6 \end{array}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0.$$

$\therefore u$ is harmonic.

11. Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$ [A.U.A/M 2017 R-13]

Soln:

Qn : $w = z^2 + \frac{1}{z^2}$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

Differentiation of (1) w.r. to z , we get

$$\frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4 - 2}{z^3}$$

Case (i)

$$\frac{dw}{dz} = 0 \Rightarrow \frac{2z^4 - 2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 = 1$$

$$\Rightarrow z^4 = 1$$

$$z = \pm 1, \pm i$$

Case (ii)

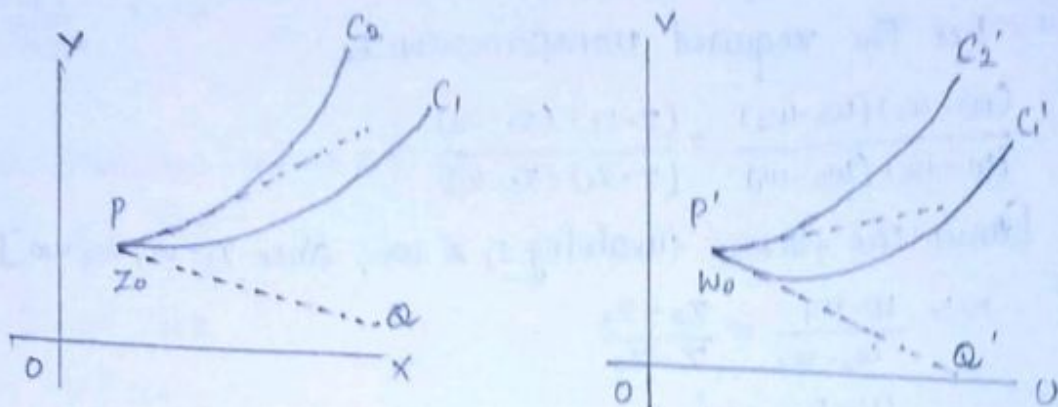
$$\frac{dz}{dw} = 0 \Rightarrow \frac{z^3}{2z^4 - 2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0.$$

$\therefore$ The critical points are $\pm 1, \pm i, 0$.

12. Define conformal transformation. [A.U.M/Y 2006, A/M 2004]

Consider the transformation $w = f(z)$, where $f(z)$ is a single valued function of z , under this transformation, a point z_0 and any two curves C_1 and C_2 passing through z_0 in the z plane, will be mapped onto a point w_0 and two curves C'_1 and C'_2 in the w plane.

(14)
If the angle between C_1 & C_2 at z_0 is the same as the angle between C_1' & C_2' at w_0 , both in magnitude and sense, then the transformation $w = f(z)$ is said to be conformal at the point z_0 .



13. Find the map of the circle $|z|=3$ under the transformation $w=2z$. [A.U A/M 2011] [A.U N/D 2012] [A.U N/D 2016 R-13].

Soln:

Given $w = 2z$, $|z| = 3$.

$$\begin{aligned} |w| &= (2)|z| \\ &= (2)(3), \text{ since } |z|=3. \\ &= 6. \end{aligned}$$

Hence, the image of $|z|=3$ in the z plane is transformed into $|w|=6$, in the w plane under the transformation $w=2z$.

14. Find the image of the line $x=1$ under the transformation $w=z^2$.

Soln:

Qn. $w = z^2$

$$u+iv = (x+iy)^2 = x^2 - y^2 + i2xy.$$

Equating the real & imaginary parts, we get

$$u = x^2 - y^2 \quad \text{--- (1)} \quad v = 2xy \quad \text{--- (2)}$$

when $x=1$.

$$\text{(1)} \Rightarrow u = 1 - y^2 \quad \text{--- (3)} \Rightarrow y^2 = 1 - u.$$

$$\text{(2)} \Rightarrow v = 2y \quad \text{--- (4)}$$

$$v^2 = 4y^2 \Rightarrow v^2 = 4(1-u)$$

The image of $z=1$ is $v^2=4(1-u)$.

15. Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively [May 1997, N/D 2012, A/M 2017 R-08].

Soln:

Qn. $z_1 = \infty, z_2 = i, z_3 = 0$ & $w_1 = 0, w_2 = i, w_3 = \infty$.

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving z_1 & w_3 , since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{z-z_3}{z_2-z_3}$$

$$\frac{w-0}{i-0} = \frac{z-0}{i-0}$$

$$w = \frac{i^2}{z} = -\frac{1}{z} = \frac{0z + (-1)}{(1)z + 0} \quad [\because w = \frac{az+b}{cz+d}, ad-bc \neq 0 \text{ for}]$$

16. Define a critical point of the bilinear transformation [N/D 2003]

The point, at which the mapping $w=f(z)$ is not conformal i.e., $f'(z)=0$ is called a critical point of the mapping.

The transformation $w=f(z)$ is conformal, $z=f^{-1}(w)$ is also conformal. Its critical points occur at $\frac{dz}{dw}=0$.

Hence, the critical points of $w=f(z)$ occur at $\frac{dw}{dz}=0$ & $\frac{dz}{dw}=0$.

17. Find the fixed points under the transformation $w = \frac{2z-5}{z+4}$ [A.U.P.T.A/M 2003] [A.U.A.P. 2016 R-15].

Soln:

The fixed points are given by

$$z = \frac{2z-5}{z+4}$$

$$z^2 + 4z = 2z - 5$$

$$z^2 + 2z + 5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$z = -1+2i, -1-2i$$

18. Find the invariant point of the bilinear transformation.

a) $w = \frac{1+z}{1-z}$ b) $w = \frac{-z+1}{z+1}$

[A.U N/D 2007, N/D 2011]
N/D 2016 R83

Soln:

a) The invariant points are given by

$$z = \frac{1+z}{1-z}$$

$$z - z^2 = 1 - z$$

$$z^2 = -1$$

$$z = \pm i$$

(b) $z = \frac{-z+1}{z+1}$

$$z^2 + z = -z + 1$$

$$z^2 + 2z - 1 = 0$$

$$z = -1 \pm \sqrt{2}$$

19. Find the invariant points of $f(z) = z^2$. [M/J 2014 R-13]

Soln:

The invariant points are given by $z = w = f(z)$

(i.e.) $z = z^2$

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\Rightarrow z = 0 \quad z = 1$$

$$z = 0 \quad z = 1$$

20 Find the critical points of the transformation $w = z^2$

[A.O.N/D 2008] [J/F 2009]

Soln:

Given $w = z^2$

$$\frac{dw}{dz} = 2z$$

To find the critical points.

$$\text{Put } \frac{dw}{dz} = 0.$$

$$\text{(i.e.) } 2z = 0.$$

$$\boxed{z = 0.}$$

(16)

UNIT-IV

COMPLEX INTEGRATION.

1. State Cauchy's integral theorem [A.U N/D 2016 R-13, N/D 2015 R-13]

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C , then $\int_C f(z) dz = 0$.

2. Evaluate $\int_C \frac{z}{(z-1)^3} dz$, where C is $|z|=2$. Using Cauchy's integral formula. [May 1998]

Soln:

Wkt, Cauchy's integral formula is

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a). \quad \text{--- (1)}$$

Qn: $\int_C \frac{z dz}{(z-1)^3}$

Here, $f(z) = z$

$a=1$ is lies inside $|z|=2$.

$$(1) \Rightarrow \int_C \frac{z dz}{(z-1)^3} = \frac{2\pi i}{2!} f''(1)$$

$$= \pi i [0]$$

$$= 0.$$

$$[\because f(z) = z ; f'(z) = 1$$

$$f'(z) = 0 ; f''(1) = 0]$$

3. Evaluate $\int_C \frac{dz}{z+4}$ where C is the circle $|z|=2$ [A.U. N/D 2004]

[A.U. N/D 2007]

Soln:

$$\text{Let } f(z) = \frac{1}{z+4}$$

Here, $z = -4$ lies outside $|z|=2$.

$\therefore f(z)$ is analytic inside and on C .

$f'(z)$ is continuous inside C .

Hence, by Cauchy's integral theorem $\oint_C f(z) dz = 0$.

4. State Cauchy's integral formula.

Soln:

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if 'a' is any point within C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz, \text{ the integration around } C \text{ being taken in the positive direction.}$$

5. State Taylor's theorem.

A function $f(z)$, analytic inside a circle C with centre at a , can be expanded in the series.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \text{ to } \infty$$

which is convergent at every point inside C .

$\rightarrow \textcircled{1}$

6. Expand $f(z) = \log(1+z)$ as a Taylor's series about $z=0$ if $|z| < 1$.
[A.U.N/D 2006]

Solution:

Function	$f(z) = \log(1+z)$	$f'(z) = \frac{1}{1+z}$	$f''(z) = \frac{-1}{(1+z)^2}$	$f'''(z) = \frac{2}{(1+z)^3}$	$\dots$
Value at $z=0$	$f(0) = 0$	$f'(0) = 1$	$f''(0) = -1$	$f'''(0) = 2$	$\dots$

T.S about $z=0$

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots$$

$$f(z) = 0 + \frac{1}{1!}z + \frac{(-1)}{2!}z^2 + \frac{2}{3!}z^3 + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \text{ if } |z| < 1.$$

7. What is the nature of the singularity $z=0$ of the function $f(z) = \frac{\sin z - z}{z^3}$ [May 2001; May 1999]

Soln: Qn: $f(z) = \frac{\sin z - z}{z^3}$

The function $f(z)$ is not defined at $z=0$.

By L-Hospital's rule.

$$\lim_{z \rightarrow 0} \frac{\sin z - z}{z^3} = \lim_{z \rightarrow 0} \frac{\cos z - 1}{3z^2} = \lim_{z \rightarrow 0} \frac{-\sin z}{6z} = \frac{0}{0}$$

$$= \lim_{z \rightarrow 0} \frac{-\cos z}{6} = \frac{-1}{6} //$$

Since the limit exists and is finite, the singularity at $z=0$ is a removable singularity.

8. Classify the various types of singularities and give one example for each type. [Nov. 1996].

* Isolated singularity

Eg: $f(z) = \frac{1}{z}$.

This function is analytic everywhere except at $z=0$.
 $\therefore z=0$ is an isolated singularity.

* Removable singularity.

Eg: $f(z) = \frac{\sin z}{z}$.

* Essential singularity

Eg: $f(z) = e^{1/z}$

* Poles

Eg: $f(z) = \frac{1}{(z-4)^2(z-3)^4}$

Here, $z=4$ is a pole of order 2
 $z=3$ is a pole of order 4.

9. If 'c' is the circle $|z|=3$ and if $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz$ then $g(2)$ [A.U A/M 2018, Q-17]

Soln:

wkt, Cauchy's integral formula is

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Given: $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz.$

$$g(2) = \int_c \frac{2z^2 - z - 2}{z - 2} dz = \int_c \frac{f(z)}{z - 2} dz = 2\pi i f(2) \quad \text{--- (2)}$$

where $f(z) = 2z^2 - z - 2.$

$$f(2) = (2)(4) - 2 - 2 = 8 - 2 - 2 = 4.$$

$$\text{(2)} \Rightarrow g(2) = 2\pi i (4) = 8\pi i //$$

10. Evaluate $\int_c \frac{3z^2 + 7z + 1}{z + 1} dz$, where c is $|z| = \frac{1}{2}$.

Soln:

Let $f(z) = \frac{3z^2 + 7z + 1}{z - (-1)}$

Here $f(z) = \frac{3z^2 + 7z + 1}{z - (-1)}$

Here $z = -1$ lies outside c.

$\therefore f(z)$ is analytic inside and on c.

$f'(z)$ is continuous inside c.

Hence by Cauchy's integral theorem $\int_c f(z) dz = 0.$

11. Can the Cauchy integral theorem be applied for evaluating the following integrals? Hence evaluate these integrals.

$$\int_c e^{\sin z^2} dz.$$

[A.U M/J 2016 R-8]

Soln:

Let $f(z) = e^{\sin z^2}$, $f(z)$ is analytic inside and on C .

$f'(z)$ is continuous inside C .

Hence, by Cauchy's integral theorem $\int_C f(z) dz = 0$.

12. Evaluate $\int_C \frac{e^z}{z-1} dz$, where C is $|z+3|=1$ [A.U. AM 2017 R-08]

Soln:

Qn $\int_C \frac{e^z}{z-1} dz$, $|z+3|=1$.

Here, C is a circle with centre $(-3, 0)$ radius 1.

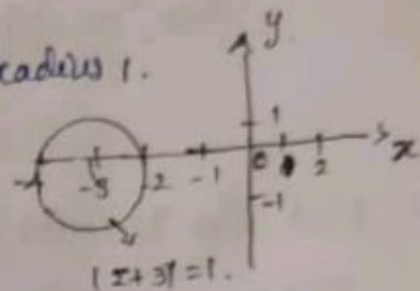
Let $f(z) = \frac{e^z}{z-1}$.

Here, $z=1$ lies outside C .

$\therefore f(z)$ is analytic inside and on C .

$f'(z)$ is continuous inside C .

Hence by Cauchy's integral theorem $\int_C f(z) dz = 0$.



13. Evaluate $\int_C \frac{e^z}{z-2} dz$, where C is the unit circle with centre as

origin.

[A.U. AM 2017 R-13]

Soln:

Given $\int_C \frac{e^z}{z-2} dz$.

Here C is the unit circle with centre as origin.

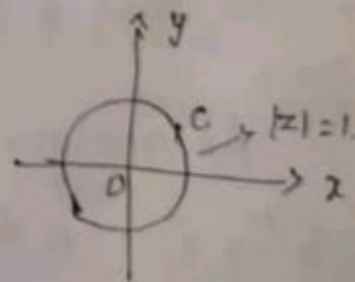
Let $f(z) = \frac{e^z}{z-2}$.

Here $z=2$ lies outside C .

$\therefore f(z)$ is analytic inside and on C .

$f'(z)$ is continuous inside C .

Hence, by Cauchy's integral theorem $\int_C f(z) dz = 0$.



14. Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Soln:

[May 2001, N/D 2009, N/D 2011]

$$Qn: f(z) = \frac{e^{2z}}{(z+1)^2}$$

Here, $z = -1$ is a pole of order 2

$$\text{wkt, } \text{Res}(z=z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

Here, $m = 2$

$$\text{Res}[z=-1] = \lim_{z \rightarrow -1} \frac{1}{1!} \frac{d}{dz} \left[(z+1)^2 \frac{e^{2z}}{(z+1)^2} \right]$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} [e^{2z}] = \lim_{z \rightarrow -1} 2e^{2z} = 2e^{-2}$$

15. Find the residues of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

[A.U M/J 2005, M/J 2012]

Soln:

Here, $z = 2$ is a simple pole

$$\text{Res}_{z=2} f(z) = \lim_{z \rightarrow 2} (z-2) \frac{4}{z^3(z-2)}$$

$$= \lim_{z \rightarrow 2} \frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}$$

16. Define poles & simple poles.

Poles:

An analytic function $f(z)$ with a singularity at $z=a$ if $\lim_{z \rightarrow a} f(z) = \infty$ then $z=a$ is a pole of $f(z)$.

Simple pole:

A pole of order one is called a simple pole.

Ex: If $f(z) = \frac{1}{(z-4)^2(z-3)^3(z-1)}$

Here, $z=1$ is a simple pole.

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$z=3$ is a pole of order 3.

$z=4$ is a pole of order 2.

17. Define Singular point:

A point $z=z_0$ at which a function $f(z)$ fails to be analytic is called a singular point (or) singularity of $f(z)$.

Ex: Consider $f(z) = \frac{1}{z-3}$.

Here, $z=3$ is a singular point of $f(z)$.

18. Find the residue of $f(z) = \frac{z^2}{(z-2)(z+1)^2}$ at $z=2$.

Soln:

$$f(z) = \frac{z^2}{(z-2)(z+1)^2}$$

Here, $z=2$ is a simple pole.

$$\text{Res } f(z) \text{ at } z=2 \Rightarrow \lim_{z \rightarrow 2} (z-2) \frac{z^2}{(z-2)(z+1)^2} = \frac{4}{9}.$$

$$\left[\because \lim_{z \rightarrow a} (z-a)^n f(z) \right]$$

19. State Cauchy's residue theorem.

If $f(z)$ be analytic at all points inside and on a simple closed curve C , except for a finite number of isolated singularities

$z_1, z_2, \dots, z_n$ inside C , then $\int_C f(z) dz = 2\pi i \sum_{i=1}^n \text{Res } f(z) \text{ at } z_i$.

$$= 2\pi i [\text{sum of the residues of } f(z) \text{ at } z_1, z_2, \dots, z_n]$$

$$= 2\pi i \sum_{i=1}^n R_i \text{ where } R_i \text{ is the residue of } f(z) \text{ at } z=z_i.$$

c 20. Essential Singularity:

If the Principal part contains an infinite number of non-zero terms, then $z=z_0$ is known as an essential singularity.

Ex:

$$f(z) = e^{1/z} = 1 + \frac{1}{1!} \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \dots$$

has $z=0$ as an essential singularity.

Since, $f(z)$ is an infinite series of negative powers of z .

$f(z) = e^{\frac{1}{z-4}}$ has $z=4$ as an essential singularity.

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UNIT-V

LAPLACE TRANSFORM.

1. State sufficient conditions for the existence of Laplace transform
[A.U. AP/May 2003, CBT J/F 2009, A/M 2011]

Soln:

- (i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$ where $a > 0$.
(ii) $f(t)$ should be of exponential order.

2. Give an example of a function such that it has Laplace transform but it is not satisfying continuity condition.
[A.U. N/D 2005] [A.U. CBT J/F 2009].

Soln: Eg: $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

3. Give two examples for which the Laplace transform do not exist.
[A.U. N/D 2007, N/D 2008]

Soln:

- (i) $L[e^{t^2}]$ does not exist since e^{t^2} is not of any exponential order.
(ii) $L[\tan t]$ does not exist since $\tan t$ is not piecewise continuous.

4. Find the Laplace transform of $f(t) = \begin{cases} t; & 0 < t < 1 \\ 0; & t > 1 \end{cases}$

[Anna April 1999; May 2000 P-T]

Soln:

$$\begin{aligned} \text{w.k.t, } L[f(t)] &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^1 e^{-st} t dt + \int_1^{\infty} e^{-st} \cdot 0 dt = \int_0^1 e^{-st} t dt \\ &= \left[t \left[\frac{e^{-st}}{-s} \right] - (1) \left[\frac{e^{-st}}{s^2} \right] \right]_0^1 \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^1 \\
&= - \left[\frac{te^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_0^1 \\
&= - \left[\left(\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \right) - \left(0 + \frac{1}{s^2} \right) \right] \\
&= \frac{1}{s^2} - e^{-s} \left[\frac{1}{s} + \frac{1}{s^2} \right] \\
&= \frac{1}{s^2} - e^{-s} \left[\frac{s+1}{s^2} \right] \\
&= \frac{1}{s^2} [1 - e^{-s}(s+1)].
\end{aligned}$$

5. State the first shifting theorem on Laplace transform.

[A.U M/J 2012]

If $L[f(t)] = F(s)$ then $L[e^{at}f(t)] = F(s-a)$.

If $L[f(t)] = F(s)$ then $L[e^{-at}f(t)] = F(s+a)$.

6. Find $f(t)$ if the Laplace transform of $f(t)$ is $\frac{s}{(s+1)^2}$

Soln:

[A.U N/D 2014 P-13]

$$\text{Given } L[f(t)] = \frac{s}{(s+1)^2}$$

$$f(t) = L^{-1} \left[\frac{s}{(s+1)^2} \right] = L^{-1} \left[\frac{s+1-1}{(s+1)^2} \right]$$

$$= L^{-1} \left[\frac{1}{s+1} - \frac{1}{(s+1)^2} \right]$$

$$= L^{-1} \left[\frac{1}{s+1} \right] - L^{-1} \left[\frac{1}{(s+1)^2} \right]$$

$$= e^{-t} - e^{-t} L^{-1} \left[\frac{1}{s^2} \right]$$

$$= e^{-t} - e^{-t} t = (1-t) e^{-t} //$$

7. Find $L[tsinat]$ [A.U. M/2008] (2)

Soln: w.k.t. $L[t^n f(t)] = (-1)^n F^{(n)}(s)$

$$\begin{aligned} L[tsinat] &= -\frac{d}{ds} [L[sinat]] \\ &= -\frac{d}{ds} \left[\frac{a}{s^2+a^2} \right] = -\left[\frac{-a(2s)}{(s^2+a^2)^2} \right] = \frac{2as}{(s^2+a^2)^2} \end{aligned}$$

8. Does Laplace transform of $\frac{\cos at}{t}$ exist? Justify. [A.U/M 2001]

Soln:

$$\lim_{t \rightarrow 0} \frac{\cos at}{t} = \frac{1}{0} = \infty$$

$\therefore L\left[\frac{\cos at}{t}\right]$ does not exist.

w.k.t,

If $L[f(t)] = F(s)$ and $\frac{1}{t} f(t)$ has a limit as $t \rightarrow 0$ then

$$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s) ds.$$

9. Find the Laplace transform of $\frac{1-\cos t}{t}$ [A.U. M/2009] [N/D 2017 P-8]

Soln:

$$L\left[\frac{1-\cos t}{t}\right]$$

$$\lim_{t \rightarrow 0} \frac{1-\cos t}{t} = \frac{0}{0}$$

$$(2) \lim_{t \rightarrow 0} \frac{\sin t}{1} = \frac{0}{1} = 0 \text{ by L'Hospital rule}$$

$\therefore L\left[\frac{1-\cos t}{t}\right]$ value exist.

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t] ds.$$

$$= \int_s^\infty [L(1) - L(\cos t)] ds.$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1^2} \right) ds = \left[\log s - \frac{1}{2} \log(s^2+1) \right]$$

$$\begin{aligned}
 &= \left[\log \left[\frac{s}{\sqrt{s^2+1}} \right] \right]_s^\infty \\
 &= 0 - \log \left[\frac{s}{\sqrt{s^2+1}} \right] \\
 &= \log \left[\frac{\sqrt{s^2+1}}{s} \right] //
 \end{aligned}$$

10. Find the Laplace transform of $\left[\frac{t}{e^t} \right]$. [A.U A/M 2018, R-17]

Soln:

$$\begin{aligned}
 L[t/e^t] &= L[t e^{-t}] \\
 &= \frac{1}{(s+1)^2} \quad \left[\because L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}} \right].
 \end{aligned}$$

11. Find $L\left[\frac{\sin t}{t}\right]$. [A.U M/M 2014 R-13, N/D 2016 R-08]

Soln:

w.k.t. $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty L[f(t)] ds.$

If $\left(1 + \frac{f(t)}{t}\right)_{t \rightarrow \infty}$ is finite $\Rightarrow \lim_{t \rightarrow 0} \left[\frac{\sin t}{t}\right] = \text{finite}.$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t) ds.$$

$$= \int_s^\infty \frac{1}{s^2+1} ds = [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s.$$

$$= \tan^{-1} \frac{1}{s}$$

$$\begin{aligned}
 &[\because \tan^{-1} s + \cot^{-1} s = \pi/2 \\
 &\quad \checkmark \cot^{-1} s = \pi/2 - \tan^{-1} s]
 \end{aligned}$$

$$[\because 1) \tan^{-1}\left(\frac{s}{a}\right) + \cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2}$$

$$3) \cot^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).]$$

$$\therefore L\left[\frac{\sin t}{t}\right] = \tan^{-1} \frac{1}{s}$$

12. State the final value theorem for Laplace transforms.
 [A.U Oct/Nov. 2002] [A.U N/D 2015 R-13]

Statement:

Final value theorem states that if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s).$$

13. Find the Laplace transform of unit step function.

Soln:

[A.U M/J 2010] [A.U N/D 2017 R-08]

The L.T of the unit step function is given by.

$$\begin{aligned} \mathcal{L}\{u(t-a)\} &= \int_0^{\infty} e^{-st} u(t-a) dt \\ &= \int_0^a e^{-st} 0 dt + \int_a^{\infty} e^{-st} (1) dt \\ &= \int_a^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_a^{\infty} = \left[\frac{e^{-\infty}}{-s} - \frac{e^{-sa}}{-s} \right] \\ &= 0 - \frac{e^{-sa}}{-s} \\ &= \frac{e^{-as}}{s} // \end{aligned}$$

14. Find $\mathcal{L}^{-1}(f)$ & Inverse Laplace transform of $\frac{e^{-as}}{s}$.

[A.U N/D 2011]

Soln:

$$\mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right] = u(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a // \end{cases}$$

$$\mathcal{L}^{-1}(f) = \delta(t).$$

15. Find $\mathcal{L}^{-1}[\cot^{-1}(s)]$ [A/M 2000] [A.U M/J 2010, A/M 2011].

Soln:

$$\text{w.k.t, } \mathcal{L}^{-1}F(s) = -\frac{1}{t} \mathcal{L}^{-1}[F'(s)]$$

$$\begin{aligned}
 \mathcal{L}^{-1}[\cot^{-1}(s)] &= -\frac{1}{t} \mathcal{L}^{-1}\left[\frac{d}{ds} \cot^{-1}s\right] \\
 &= -\frac{1}{t} \mathcal{L}^{-1}\left[\frac{-1}{1+s^2}\right] \\
 &= \frac{1}{t} \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
 &= \frac{1}{t} \sin t \quad //
 \end{aligned}$$

16. Find $\mathcal{L}^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$ [A.U N/D 2013] [A.U AP 2011 R-15 U.D].

Soln: We k.t, $\mathcal{L}^{-1}[F(s)] = -\frac{1}{t} \mathcal{L}^{-1}[F'(s)]$

$$\begin{aligned}
 \mathcal{L}^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] &= -\frac{1}{t} \mathcal{L}^{-1}\left[\frac{d}{ds} \left[\log\left(\frac{s+1}{s-1}\right)\right]\right] \\
 &= -\frac{1}{t} \mathcal{L}^{-1}\left[\frac{d}{ds} \log(s+1) - \log(s-1)\right] \\
 &= -\frac{1}{t} \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] \\
 &= -\frac{1}{t} [e^{-t} - e^t] \\
 &= \frac{1}{t} [e^t - e^{-t}] \\
 &= \frac{2}{t} \left[\frac{e^t - e^{-t}}{2}\right] \\
 &= \frac{2}{t} \sinh t.
 \end{aligned}$$

17. Find $\mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+2)^{5/2}}\right]$

[A.U N/D 2015]

Soln:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^{5/2}}\right] = e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s^{5/2}}\right]$$

$$\begin{aligned}
 \therefore \mathcal{L}[t^n e^{-at}] &= \frac{n!}{(s+a)^{n+1}} \\
 \frac{t^n e^{-at}}{n!} &= \mathcal{L}^{-1}\left[\frac{1}{(s+a)^{n+1}}\right]
 \end{aligned}$$

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$$= e^{-2t} \frac{4}{3\sqrt{\pi}} t^{5/2}$$

$$[\because L^{-1}[e^{-as}F(s)] = f(t-a)U(t-a)]$$

$$L^{-1}\left[\frac{e^{-s}}{(s+2)^{5/2}}\right] = \frac{4}{3\sqrt{\pi}} (t-1)^{5/2} e^{-2(t-1)} U(t-1)$$

18. State convolution theorem of L.T. [A/V March 1995, N/D 2003, M/J 2005]

Statement:

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$ then

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$$

19. Using convolution theorem find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ [A/V Nov. 1996, Mar 1996, M/M 2017, A/M 08, N/D 2007, M/J 2012, N/D 2013]

Soln:

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * \frac{1}{a} L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \cos at * \frac{1}{a} \sin at$$

$$= \frac{1}{a} [\cos at * \sin at]$$

$$= \frac{1}{a} \int_0^t \cos au \sin a(t-u) du$$

$$= \frac{1}{a} \int_0^t \sin(at-au) \cos a u du$$

$$= \frac{1}{a} \int_0^t \frac{\sin(at-au+au) + \sin(at-au-au)}{2} du$$

$$[\because \sin a \cos b = \frac{1}{2} \sin(a+b) + \sin(a-b)]$$

$$\begin{aligned}
&= \frac{1}{2a} \int_0^t [\sin at + \sin a(t-2u)] du \\
&= \frac{1}{2a} \left[(\sin at)u + \left(-\frac{\cos a(t-2u)}{-2a} \right) \right]_0^t \\
&= \frac{1}{2a} \left[u \sin at + \frac{\cos a(t-2u)}{2a} \right]_0^t \\
&= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - \left(0 + \frac{\cos at}{2a} \right) \right] \\
&= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a} \right] \\
&= \frac{1}{2a} t \sin at \quad //
\end{aligned}$$

20. Solve $y'' + y = 0$ given that $y(0) = 1$, $y'(0) = 0$ using Laplace transform.
[April 2001 P-7]

Soln:

Given $y'' + y = 0$; $y(0) = 1$, $y'(0) = 0$.

$$y''(t) + y(t) = 0.$$

$$L[y''(t)] + L[y(t)] = 0.$$

$$s^2 L[y(t)] - sy(0) - y'(0) + L[y(t)] = 0.$$

$$[s^2 + 1] L[y(t)] - s = 0.$$

$$[s^2 + 1] L[y(t)] = s.$$

$$L[y(t)] = \frac{s}{s^2 + 1}.$$

$$y(t) = L^{-1} \left[\frac{s}{s^2 + 1} \right].$$

$$= \cos t.$$